

Augmenting a Geometric Matching is NP -complete

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Abstract

Given $2n$ points in the plane, it is well-known that there always exists a perfect straight-line non-crossing matching. We show that it is NP -complete to decide if a partial matching can be augmented to a perfect one, via a reduction from 1-in-3-SAT. This result also holds for bichromatic matchings.

In a blitz chess tournament every player wants to play each round without pause and play against every other player exactly once. To design such a tournament in mathematical terms means to partition the edges of the complete graph into perfect matchings. Matchings appear in everyday life and are long studied mathematical objects. They give rise to many natural mathematical questions and have many applications.

A well known fact is that any point set in general position in the plane with an even number of points admits a perfect matching [2]. To see this go for instance by induction: Sweep a vertical line ℓ from left to right till 2 points are left of ℓ , connect these 2 points and continue. Or allow crossings and consider the matching with smallest total edge-length. This settles also the bicolored case.

When one works on any kind of problem closely related to geometric matchings, one might take a pen and a piece of paper and start drawing dots representing points in the plane; just to get an intuition on whatever problem one is thinking about. One might continue to draw edges arbitrarily in a not too simple way, in the hope to see something or get a new idea. But then one realizes that there are only few points left and one has problems to continue adding edges without introducing crossings. Of course one might cheat and just draw more points at appropriate places. Sometimes one must do that. See Figure 1.

Because matchings are so natural and important mathematical objects, we were motivated to study the augmentability problem. For related augmenting problems we refer to a recent survey from Ferran Hurtado and Csaba D. Tóth

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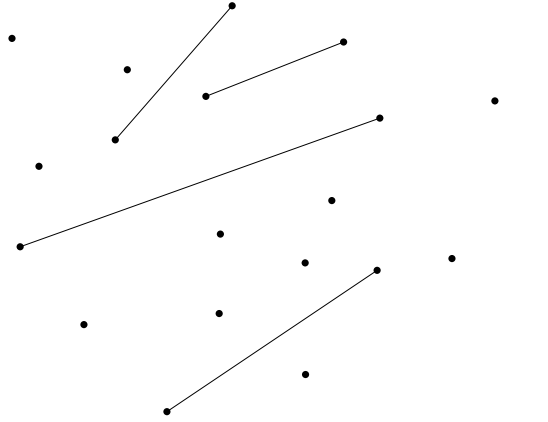


Figure 1: After some edges are drawn it is not clear, how to continue.

[1]. After we present the proof of our main Theorem 1, we will present a simple application in Theorem 2, to illustrate its versatility. Theorem 2 has already been shown by Alexander Pilz [3].

To avoid confusions we repeat some definitions. A *PSLG* or *planar straight line graph* consists of points and line segments in the plane representing vertices and edges of an abstract graph. A PSLG is a *partial (geometric) matching* if each vertex is incident to at most one edge. In a *perfect matching* every vertex is incident to exactly one edge. We say a PSLG is *bichromatic* if every vertex has color *red* or *blue* and no two vertices with the same color have a common edge. In this paper we say that a PSLG $G = (V, E)$ augments $G' = (V', E')$ if they have the same vertex set and $E' \subseteq E$. A PSLG is *cubic* if every vertex is incident to exactly 3 edges.

Theorem 1 (Augmenting Matchings). *It is NP-complete to decide whether a partial geometric matching can be augmented to a perfect one, both for the colored and uncolored case.*

Proof: We will prove the colored case first and show later how the proof carries over to the uncolored case. We will reduce 1-in-3-SAT to the decision problem above. It can be checked in linear time whether a given set of edges augments a matching. This is true for the colored and uncolored case. Thus the decision problem lies in *NP*. We will first describe gadgets and then describe how to encode an instance of 1-in-3-SAT in an instance of our augmentability problem using these gadgets [4].

The construction not surprisingly consists of *variable gadgets* and *clause gadgets*. We will use *lanes* to transport the information from the variable gadgets to the clause gadgets. Lanes might have to cross each other on their way¹ from the variable gadget to the clause gadget and at other points which will become

¹Even when we considered a reduction from *planar* 1-in-3-SAT [4], junctions were still necessary.

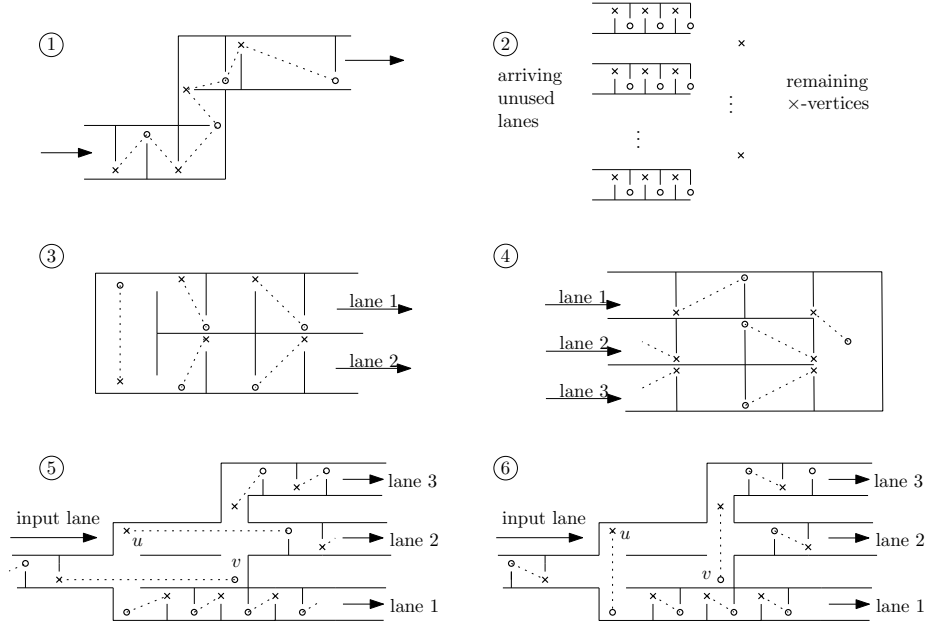


Figure 2: 1: The lane surrounds the free vertices such that each vertex has only two possible neighbors it can connect to; 2: The basin consists of all the lanes that we did not use. 3: The variable gadget consists of 2 free vertices of different color, each emits one lane; 4: The clause gadget consists of exactly one \circ -vertex which can be matched to exactly one of the arriving lanes; 5&6: The multiplier has one lane arriving from the left and three lanes emit. Depending on the information, the lane carries, there is only one way to match the edges. In each case lane 1 carries the opposite information as the input lane and lanes 2 and 3 carry the same information as the input lane.

apparent later. For this purpose *junction gadgets* or just *junctions* will be introduced. To encode the negation of a variable, we have certain *multiplier-gadgets* or just *multipliers*, which also serve for us to split a lane in two. This is necessary as our variable gadgets only emit two lanes initially. Thus, we can create as many lanes as we want with the multiplier gadget to transport the information from a certain variable. This is necessary if a variable appears in more than one clause of the original formula. In the end we will have the problem that some lanes will not be used. We let all of them go into a common *basin*, where they can be taken care of easily. See Figure 2 and Figure 3 for illustrations. All of our gadgets consist of the input edges and free vertices arranged in the plane. For the clarity of the drawings we did not draw the edges separately, but let them appear as one PSLG. This is not problematic as one could perturb each edge slightly such that they do not overlap nor cross and still all the crucial properties remain.

We start to describe the lane, which transports information. The lane consists of a tube bounded by matching edges and unused points. The tube is piked with matching edges to guarantee that each free vertex has exactly two possible neighbors he could be connected to. Thus any matching edge determines the matching of all edges in the lane and there are exactly two ways to match all the free vertices. Also we attach to a lane a direction as we say it starts at a variable gadget and ends in a clause gadget or the basin; maybe bypassing several other gadgets in between.

We say that a lane transports the information *true* if the orientation of the edge is $\times\circ$ (with respect to the direction of the lane) and *false* if it is $\circ\times$.

Note that lanes do not need to be straight, but can have any kind of bends.

The variable gadget consists of 2 free-vertices at the start of 2 lanes. We say that a variable is set to *true* if the two free vertices are matched to one another, and *false* otherwise. The upper lane transports the information x and the lower one transports the information $\neg x$ by our convention.

The basin is in fact not a real gadget as we only gather all the lanes to a common location and place sufficiently many \times -vertices next to these lanes to serve them (i.e. connect the \circ -vertices, that are not connected yet.). It holds $\#(\times\text{-vertices in the basin}) = \#(\circ\text{-vertices}) - \#(\times\text{-vertices})$. This is obvious as the total number of \times -vertices and \circ -vertices must agree.

The clause gadget consists of exactly one free \circ -vertex which is exposed to three arriving lanes. The \circ -vertex can only be matched to exactly one of the three lanes. Thus all free vertices can be matched iff exactly one of the lanes transports the information *true*.

The multiplier gadget accepts one arriving lane and emits 3 leaving ones. Lane 1 and 2 as shown in Figure 2 carry the same information as the arriving lane and lane 3 will transport the opposite information. Observe that the vertex denoted v in Figure 2 can only be adjacent to two vertices. To which is determined by the information of the input lane. This in turn determines what u is connected to and will decide the edges for the three emitting lanes.

The junction is depicted in Figure 3 together with all possible assignments of the incoming lanes and a notation for the vertices. Depending on lane 2 the point v must be matched with u or w . This implies that either the left tunnel or the right tunnel is blocked by an edge. This is no problem when lane 1 carries the *true* information, as in the top of Figure 3. But it is still no problem when lane 1 carries the information *false*. At vertex r has to be made a choice such that the information can be passed through the correct tunnel, as in the bottom of Figure 3.

We are now ready to present the whole construction, see Figure 4 for an illustration. The construction consist of 5 parts. The first part consists merely of variable gadgets. The second part populates the lanes from the variables using the multiplier. This is done till each literal is at least as many times present as it is used in the clauses. The third part is the most important one, here the literals are connected to the corresponding clauses. Note that we have at most a quadratic number of crossings. At the bottom of the third part is the fourth part, the basin, which takes care of overmuch lanes. The last part are

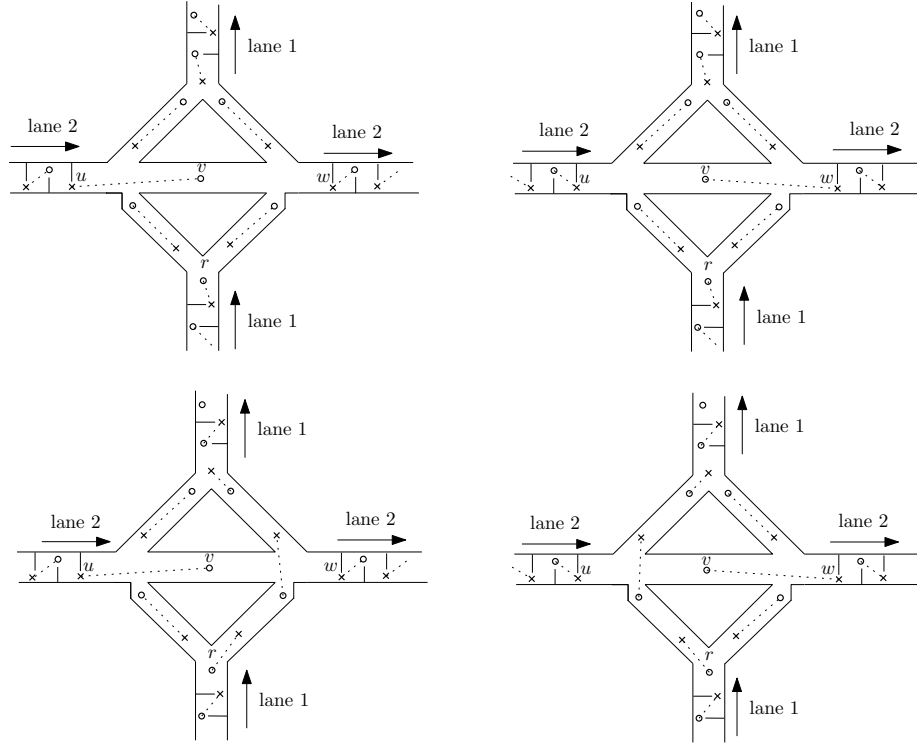


Figure 3: The junction is needed to let two lanes cross. There are four possible combinations of information the lanes carry. For each combination there is only one way to match the free vertices and the information is always carried over to the other side of the junction.

the clause gadgets, where the lanes end. It is clear, that this construction can be made in polynomial time. Now let Φ be an instance of 1-in-3-SAT and $M(\Phi)$ the construction presented. If there is an assignment to the variables such that Φ is true we can match the variable gadget accordingly and transport these information over the lanes and each clause gadget will be satisfied. Thus it is possible to augment the matching of $M(\Phi)$ to a perfect one. On the other hand if $M(\Phi)$ can be augmented to a perfect matching we get an assignment from the variable gadget, which satisfies each clause.

Note that we have not used the color restriction in the proof. This implies that each gadget would still work if the colors were removed and thus it is also *NP*-hard to decide whether a partial matching of an uncolored point set can be augmented to a perfect one. This completes the proof. ☺

As promised we present now a new proof of a recent result from Alexander Pilz using Theorem 1.

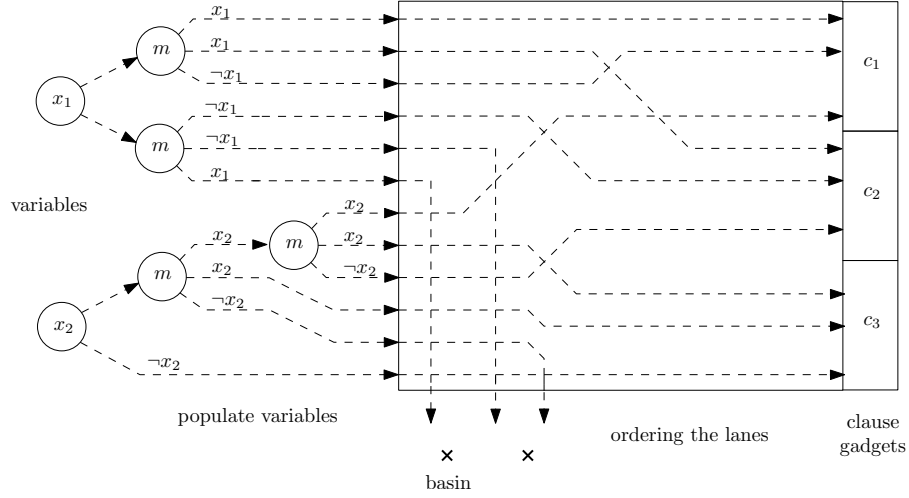


Figure 4: The construction: lanes are depicted by dashed lines; the multipliers are represented by a circle with an m ; variable gadgets are represented by the corresponding variable names surrounded by a circle; when two lanes cross a junction is implied

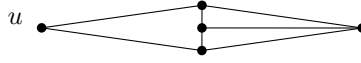


Figure 5: All vertices of this graph except u have 3 incident edges

Theorem 2 (Augmenting cubic PSLG, Pilz [3]). *It is NP-complete to decide whether a PSLG can be augmented to a cubic one.*

Proof: Clearly the problem lies in *NP*. To see that it is *NP*-hard we make a reduction from the previous problem. Let M be a partial matching. Replace every vertex v by a small rotated copy of the gadget in Figure 5. The edges of M remain where they were. We denote the resulting PSLG by $G(M)$.

It is clear that this can be done in polynomial time and we observe that M can be augmented to a perfect matching iff $G(M)$ can be augmented to a cubic PSLG. ☺

Acknowledgments

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